

POLYNOMIAL DYNAMICAL SYSTEMS AND KORTEWEG–DE VRIES EQUATION

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ABSTRACT. In this work we explicitly construct polynomial vector fields \mathcal{L}_k , $k = 0, 1, 2, 3, 4, 6$ on the complex linear space \mathbb{C}^6 with coordinates $X = (x_2, x_3, x_4)$ and $Z = (z_4, z_5, z_6)$. The fields \mathcal{L}_k are linearly independent outside their discriminant variety $\Delta \subset \mathbb{C}^6$ and tangent to this variety.

We describe a polynomial Lie algebra of the fields \mathcal{L}_k and the structure of the polynomial ring $\mathbb{C}[X, Z]$ as a graded module with two generators x_2 and z_4 over this algebra. The fields \mathcal{L}_1 and \mathcal{L}_3 commute. Any polynomial $P(X, Z) \in \mathbb{C}[X, Z]$ determines a hyperelliptic function $P(X, Z)(u_1, u_3)$ of genus 2, where u_1 and u_3 are coordinates of trajectories of the fields \mathcal{L}_1 and \mathcal{L}_3 .

The function $2x_2(u_1, u_3)$ is a 2-zone solution of the KdV hierarchy and $\frac{\partial}{\partial u_1} z_4(u_1, u_3) = \frac{\partial}{\partial u_3} x_2(u_1, u_3)$.

INTRODUCTION

Consider the hyperelliptic curve of genus g

$$V_\lambda = \left\{ (x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1} + \sum_{k=2}^{2g+1} \lambda_{2k} x^{2g-k+1} \right\}. \quad (1)$$

For brevity, we call Abelian functions on Jacobi varieties of (1) the hyperelliptic functions of genus g . In the theory and applications of these functions, that are based on the sigma-function $\sigma(u; \lambda)$ (see [2, 5, 6, 12]), where $u = (u_1, u_3, \dots, u_{2g-1})$, $\lambda = (\lambda_4, \dots, \lambda_{4g+2})$, the grading plays an important role. Further we use an indexing of variables, parameters and functions that clearly indicates their grading. In our notation, for $\omega = (j_1, \dots, (2g-1)j_g)$, where $j_1 \geq 0, \dots, j_g \geq 0$ and $j_1 + \dots + j_g \geq 2$, we have

$$\wp_\omega(u; \lambda) = -\frac{\partial^{j_1+\dots+j_g}}{\partial u_1^{j_1} \dots \partial u_g^{j_g}} \ln \sigma(u; \lambda),$$

$\deg \wp_\omega = j_1 + \dots + (2g-1)j_g$.

According to Dubrovin–Novikov theorem [15], the space of the universal bundle of Jacobi varieties of hyperelliptic curves (1) is birationally equivalent to the complex linear space \mathbb{C}^{3g} . A proof of this result based on the theory of hyperelliptic sigma-functions $\sigma(u; \lambda)$ was obtained in [6, 12] and is essentially used in this work.

In [5] (see also [6, 12]) it is shown that for genus $g > 1$ the hyperelliptic analogue $\wp_{2,0,\dots,0}(u; \lambda)$ of Weierstrass elliptic function $\wp(u; g_2, g_3) = \wp_2(u; -4\lambda_4, -4\lambda_6)$ gives a family of g -zone solutions of Korteweg–de Vries equation (KdV) introduced in [19].

In [11] the classical problem of differentiation of Abelian functions over parameters for families of (n, s) -curves was solved. In the case of elliptic curves (with $(n, s) = (2, 3)$) this problem was solved by Frobenius and Stickelberger in [16].

In [11] in the case of hyperelliptic curves $((n, s) = (2, 2g+1))$ a method of construction of $3g$ polynomial dynamical systems on \mathbb{C}^{3g} was developed. These systems are determined

This work is supported by the Russian Science Foundation under grant 14-50-00005.

by polynomial vector fields that are linearly independent outside their discriminant variety and tangent to this variety.

In the differential field of hyperelliptic functions of genus g there are the generators $\wp_{\omega_k}(u; \lambda)$, $k = 1, \dots, g$, where $\omega_1 = (2, 0, \dots, 0)$, $\omega_2 = (1, 3, 0, \dots, 0)$, \dots , $\omega_g = (1, 0, \dots, 0, 2g-1)$. The function $2\wp_{\omega_1}(u; \lambda)$ is a g -zone solution of the KdV hierarchy and $\frac{\partial}{\partial u_1}\wp_{\omega_k} = \frac{\partial}{\partial u_{2k-1}}\wp_{\omega_1}$, $k = 2, \dots, g$ (see [5, 6, 12]).

Problem. Construct polynomial vector fields \mathcal{L}_{2k-2} , $k = 1, \dots, 2g$, and \mathcal{L}_{2q-1} , $q = 1, \dots, g$, on \mathbb{C}^{3g} with coordinates $X_1 = (x_{2,0}, x_{3,0}, x_{4,0}), \dots, X_g = (x_{1,2g-1}, x_{2,2g-1}, x_{3,2g-1})$ and the corresponding polynomial Lie algebra \mathcal{A}_g , such that:

- 1) The ring of polynomials $\mathbb{C}[X_1, \dots, X_g]$ is a module with g generators $x_{2,0}, x_{1,3}, \dots, x_{1,2g-1}$ over the Lie algebra \mathcal{A}_g .
- 2) The fields $\mathcal{L}_1, \mathcal{L}_3, \dots, \mathcal{L}_{2g-1}$ commute.
- 3) Any polynomial $P(X_1, \dots, X_g) \in \mathbb{C}[X_1, \dots, X_g]$ determines a hyperelliptic function $P(X_1, \dots, X_g)(u)$ of genus g , where $u = (u_1, u_3, \dots, u_{2g-1})$ are coordinates of trajectories of the fields $\mathcal{L}_1, \mathcal{L}_3, \dots, \mathcal{L}_{2g-1}$.
- 4) The generators of the \mathcal{A}_g -module determine the generators $x_{2,0}(u) = \wp_{\omega_1}(u)$, $x_{1,2k-1}(u) = \wp_{\omega_k}(u)$, $k = 2, \dots, g$, of the differential field of hyperelliptic functions of genus g .

In this paper, this problem is solved in the case $g = 2$.

In §1 and §2 for the classical example of the case $g = 1$ our approach and methods are described in detail. Homogeneous polynomial dynamical systems on \mathbb{C}^3 are described in terms of universal bundles with non-singular elliptic curves in standard Weierstrass model as fiber and in terms of universal bundles with factor of non-degenerate elliptic curves by canonical involution as fiber. The main results are given in §3 and §4. Homogeneous polynomial dynamical systems on \mathbb{C}^6 are described in terms of universal bundles with Jacobi varieties of non-singular hyperelliptic curves of genus 2 as fiber and Kummer varieties of this curves as fiber.

In Appendices A and B we collected the necessary information about elliptic sigma-functions and sigma-functions of genus 2 curves.

Note that in general, the present work belongs to a large field of research at the junction of the analytic theory of Riemann surfaces, Abelian functions, and the theory of integrable systems. One of the main objects of this field are infinite-dimensional Lie algebras, which are finite-dimensional modules over the corresponding rings of functions. As recent publications in this area, see [20, 21, 22].

The author is grateful to E. Yu. Bunkova, O. I. Mokhov and V. Z. Enolskii for a fruitful discussion of the results, and to E. Yu. Bunkova for great help in preparing this work for publication.

1. POLYNOMIAL DYNAMICAL SYSTEMS ON \mathbb{C}^3

Consider the bundle

$$\pi: U_1 \longrightarrow B_1$$

of Jacobi varieties on non-singular elliptic curves

$$V_g = \{(x, y) \in \mathbb{C}^2 : y^2 = 4x^3 - g_2x - g_3\} \quad (2)$$

with base $B_1 = \{(g_2, g_3) \in \mathbb{C}^2 : \Delta(g_2, g_3) \neq 0\}$, where $\Delta(g_2, g_3) = g_2^3 - 27g_3^2$, and fiber $J_g = \mathbb{C}^1/\Gamma_g$, where Γ_g is a lattice of rank 2, generated by the periods $(2\omega, 2\omega')$ of a holomorphic differential $\frac{dx}{y}$ on cycles of the curve V_g . Denote by \mathcal{F} the field of C^∞ -functions on U_1 , such that their restrictions to the layers J_g are elliptic functions. For any point $g = (g_2, g_3) \in B_1$ on the universal covering $\mathbb{C}^1 \rightarrow J_g$ a Weierstrass sigma-function

$\sigma(u; g_2, g_3)$ is defined (see Appendix A). It determines Weierstrass functions [23, 25]

$$\zeta(u; g_2, g_3) = \frac{\partial \ln \sigma(u; g_2, g_3)}{\partial u} \quad \text{and} \quad \wp(u; g_2, g_3) = -\frac{\partial \zeta(u; g_2, g_3)}{\partial u}.$$

Any elliptic function on the Jacobi variety J_g is a rational function in $\wp(u; g_2, g_3)$ and $\frac{\partial}{\partial u} \wp(u; g_2, g_3)$. Denote by $\text{Der}(\mathcal{F})$ the \mathcal{F} -module of derivations of the field \mathcal{F} . According to Frobenius–Stickelberger theorem (see [16]) the \mathcal{F} -module $\text{Der}(\mathcal{F})$ is generated by the operators

$$L_0 = -u\partial_u + 4g_2\partial_{g_2} + 6g_3\partial_{g_3}, \quad L_2 = -\zeta(u; g_2, g_3)\partial_u + 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3} \quad \text{and} \quad L_1 = \partial_u.$$

In [11] a proof of this result is given. It uses the operators Q_0 and Q_2 (see Appendix A) annihilating sigma-functions $\sigma(u; g_2, g_3)$.

The operators Q_0 and Q_2 were discovered by Weierstrass after the work [16]. The application of operators Q_0 and Q_2 made it possible to considerably simplify the process of obtaining the operators L_0 , L_2 and L_1 (see [11]).

Consider the complex linear space \mathbb{C}^3 with graded coordinates x_2, x_3, x_4 , $\deg x_k = k$, and the space \mathbb{C}^2 with graded coordinates g_2, g_3 , $\deg g_k = 2k$. We introduce the homogeneous polynomial map

$$\pi_1: \mathbb{C}^3 \rightarrow \mathbb{C}^2 : \pi_1(x_2, x_3, x_4) = (g_2, g_3),$$

where $\pi_1^*(g_2) = 12x_2^2 - 2x_4$, $\pi_1^*(g_3) = -8x_2^3 + 2x_4x_2 - x_3^2$. The Jacobi matrix of π_1 has the form

$$\begin{pmatrix} 24x_2 & 0 & -2 \\ -24x_2^2 + 2x_4 & -2x_3 & 2x_2 \end{pmatrix}.$$

Hence the point $X = (x_2, x_3, x_4)$ is singular if and only if

$$x_2x_3 = 0, \quad x_3 = 0, \quad x_4 = 0.$$

We have

$$\pi_1(x_2, 0, 0) = (12x_2^2, -8x_2^3) \in \Delta.$$

Set

$$\widetilde{\Delta} = \{X = (x_2, x_3, x_4) \in \mathbb{C}^3 \mid \pi_1(X) \in \Delta\} \quad \text{and} \quad \widetilde{W}_1 = \mathbb{C}^3 \setminus \widetilde{\Delta}.$$

This defines the regular bundle

$$\pi_1: \widetilde{W}_1 \rightarrow B_1.$$

It's fiber over the point $g = (g_2, g_3)$ is the non-singular curve

$$x_3^2 = 4x_2^3 - \pi_1^*(g_2)x_2 - \pi_1^*(g_3).$$

In the space $\widetilde{W}_1 \subset \mathbb{C}^3$ the involution $(x_2, x_3, x_4) \rightarrow (x_2, -x_3, x_4)$ acts freely, so the regular bundle π_1 decomposes into a regular double covering $\widetilde{W}_1 \rightarrow \mathcal{K}_1$ and a regular bundle $\mathcal{K}_1 \rightarrow B_1$ with fiber corresponding to factor of an elliptic curve by canonical involution τ_1 .

Set

$$\widetilde{U}_1 = \{(u; g_2, g_3) \in U_1 : u \neq 0\} \quad \text{and} \quad \widehat{U}_1 = \widetilde{U}_1 / (u \sim -u).$$

By the Weierstrass theorem on elliptic curve uniformization (see Appendix A) the map

$$\varphi_g: \mathbb{C} \setminus 0 \rightarrow V_g : \varphi_g(u) = (\wp(u), \wp'(u))$$

is a homeomorphism for any point $g \in B_1$.

The map φ_g is equivariant under the involution $u \rightarrow -u$ on \mathbb{C} and τ_1 on V_g . The map $u \rightarrow \wp(u)$ determines a homeomorphism of quotient-spaces.

Corollary 1 (Uniformization). *The map*

$$\varphi: \widetilde{U}_1 \rightarrow \widetilde{W}_1 : \varphi(u; g_2, g_3) = (x_2, x_3, x_4),$$

where $x_2 = \wp(u; g_2, g_3)$, $x_3 = \wp'(u; g_2, g_3)$, $x_4 = \wp''(u; g_2, g_3)$ and $f' = \frac{\partial}{\partial u}f$, determines a fiberwise equivariant with respect to involutions diffeomorphism of bundles

$$\begin{array}{ccc} \widetilde{U}_1 & \longrightarrow & \widetilde{W}_1 \\ \downarrow & & \downarrow \\ B_1 & \xlongequal{\quad} & B_1 \end{array}.$$

We introduce the following derivations of the polynomial ring on the space \mathbb{C}^3 with coordinates x_2, x_3, x_4 :

$$\mathcal{L}_0 = 2x_2\partial_{x_2} + 3x_3\partial_{x_3} + 4x_4\partial_{x_4}, \quad \mathcal{L}_2 = \frac{2}{3}(x_4 - 3x_2^2)\partial_{x_2} + 3x_2x_3\partial_{x_3} + (3x_3^2 + 2x_2x_4)\partial_{x_4},$$

$$\mathcal{L}_1 = x_3\partial_{x_2} + x_4\partial_{x_3} + 12x_2x_3\partial_{x_4}.$$

The polynomial fields $\mathcal{L}_0, \mathcal{L}_2$ and \mathcal{L}_1 can be written down using matrix

$$\mathcal{T}_1 = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ x_3 & x_4 & 12x_2x_3 \\ \frac{2}{3}x_4 - 2x_2^2 & 3x_2x_3 & 3x_3^2 + 2x_2x_4 \end{pmatrix}.$$

We have

$$3 \det \mathcal{T}_1 = -432x_2^3x_3^2 + 36x_2^2x_4^2 + 108x_2x_3^2x_4 - 27x_3^4 - 8x_4^3 = \widehat{g}_2^3 - 27\widehat{g}_3^2.$$

Here $\widehat{g}_2 = \pi_1^*(g_2)$, $\widehat{g}_3 = \pi_1^*(g_3)$,

$$\mathcal{L}_0 \det \mathcal{T}_1 = 12 \det \mathcal{T}_1, \quad \mathcal{L}_2 \det \mathcal{T}_1 = 0, \quad \mathcal{L}_1 \det \mathcal{T}_1 = 0.$$

Thus, the fields $\mathcal{L}_0, \mathcal{L}_2$ and \mathcal{L}_1 are tangent to the manifold $\{\det \mathcal{T}_1 = 0\}$.

Theorem 2 (see [11]). *The uniformizing diffeomorphism φ transforms the vector fields L_k into \mathcal{L}_k , $k = 0, 1, 2$, i.e.*

$$L_k \varphi^* P = \varphi^* \mathcal{L}_k P$$

for any polynomial $P = P(x_2, x_3, x_4)$.

Denote by τ_k the coordinate along the trajectory of the field \mathcal{L}_k , i.e.

$$\frac{\partial}{\partial \tau_k} P = L_k P, \quad k = 0, 1, 2.$$

Corollary 3. *The fields \mathcal{L}_k determine on \mathbb{C}^3 the graded homogeneous polynomial dynamical systems*

$$\begin{aligned} S_0 : \quad & \frac{\partial}{\partial \tau_0} x_2 = 2x_2; \quad \frac{\partial}{\partial \tau_0} x_3 = 3x_3; \quad \frac{\partial}{\partial \tau_0} x_4 = 4x_4, \\ S_1 : \quad & \frac{\partial}{\partial \tau_1} x_2 = x_3; \quad \frac{\partial}{\partial \tau_1} x_3 = x_4; \quad \frac{\partial}{\partial \tau_1} x_4 = 12x_2x_3, \\ S_2 : \quad & \frac{\partial}{\partial \tau_2} x_2 = \frac{2}{3}x_4 - 2x_2^2; \quad \frac{\partial}{\partial \tau_2} x_3 = 3x_2x_3; \quad \frac{\partial}{\partial \tau_2} x_4 = 3x_3^2 + 2x_2x_4. \end{aligned}$$

Corollary 4. *The fields \mathcal{L}_0 and \mathcal{L}_2 determine on \mathbb{C}^3 with coordinates (x_2, x_4, x_6) the graded homogeneous polynomial dynamical systems*

$$\begin{aligned}\widehat{S}_0 : \frac{\partial}{\partial \widehat{\tau}_0} x_2 &= 2x_2; \quad \frac{\partial}{\partial \widehat{\tau}_0} x_4 = 4x_4; \quad \frac{\partial}{\partial \widehat{\tau}_0} x_6 = 6x_6, \\ \widehat{S}_2 : \frac{\partial}{\partial \widehat{\tau}_2} x_2 &= \frac{2}{3}x_4 - 2x_2^2; \quad \frac{\partial}{\partial \widehat{\tau}_2} x_4 = 3x_6 + 2x_2x_4; \quad \frac{\partial}{\partial \widehat{\tau}_2} x_6 = 6x_2x_6.\end{aligned}$$

Set

$$\mathbb{C}_*^3 = \{(x_2, x_3, x_4) \in \mathbb{C}^3 : x_3 \neq 0\}, \quad \mathbb{C}_{*,*}^3 = \{(x_2, x_4, x_6) \in \mathbb{C}^3 : x_6 \neq 0\}.$$

The regular double covering

$$\mathbb{C}_*^3 \rightarrow \mathbb{C}_{*,*}^3 : (x_2, x_3, x_4) \rightarrow (x_2, x_4, x_3^2)$$

transforms the systems S_0 and S_2 into systems \widehat{S}_0 and \widehat{S}_2 .

Using the commutation rules for the operators L_k , $k = 0, 1, 2$, (see [11] and Appendix A) and the notion of polynomial Lie algebra (see [7]), we obtain:

Corollary 5. *There is a graded polynomial Lie algebra over the ring $\mathbb{C}[x_2, x_3, x_4]$ with generators $\mathcal{L}_0, \mathcal{L}_2$ and \mathcal{L}_1 . The action of this operators on $\mathbb{C}[x_2, x_3, x_4]$ is determined by the dynamical systems described above and the commutation relations:*

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k, \quad k = 0, 1, 2, \quad \text{and} \quad [\mathcal{L}_1, \mathcal{L}_2] = x_2\mathcal{L}_1.$$

2. DIFFERENTIAL EQUATIONS ON THE DIFFERENTIAL FIELD GENERATOR

Set $\frac{\partial}{\partial \tau_1} f = f'$. Then according to system S_1 the function $x_2 = x_2(\tau_0, \tau_1, \tau_2)$ satisfies the differential equation

$$x_2''' = 12x_2x_2'.$$

Since $g'_k(x_2, x_3, x_4) = 0$, $k = 2, 3$, we obtain the classical result:

Corollary 6. *1. The function*

$$U(u) = 2x_2 = 2\wp(u; g_2, g_3),$$

determines a two-parametric family of elliptic solutions for the stationary KdV equation

$$U''' = 6UU'.$$

2. The solution $U(u)$ of the differential equation with constant a

$$U'' = 3U^2 + a$$

is the elliptic function $U(u) = 2\wp(u; g_2, g_3)$, where $g_2 = -a$.

The functions $g_k = g_k(\tau_0, \tau_2)$, $k = 2, 3$, change while moving along the trajectories of fields \mathcal{L}_0 and \mathcal{L}_2 in such a way that the function $U(u; g_2, g_3)$ remains elliptic.

Problem. Describe the dependence of the family of solutions $\wp(u; g_2, g_3)$ of the stationary KdV equation on the variation of parameters.

We have:

$$\frac{\partial}{\partial \tau_0} x_2 = 2x_2.$$

Therefore, $x_2 = ce^{2\tau_0}$. Choose a value $\tau_{0,*}$ for which the functions $g_k(\tau_0, \tau_2)$, $k = 2, 3$, are regular, and set $g_{k,*}(\tau_2) = g_k(\tau_{0,*}, \tau_2)$. We obtain

$$\wp(u; g_2, g_3) = \wp(u; g_{2,*}(\tau_2), g_{3,*}(\tau_2))e^{2(\tau_0 - \tau_{0,*})}.$$

We have

$$\frac{\partial}{\partial \tau_2} x_2 = \frac{2}{3} x_4 - 2x_2^2.$$

Set $\tilde{\mathcal{L}}_2 = \mathcal{L}_2 + x_2 \mathcal{L}_0$ and $\tilde{\mathcal{L}}_2 = \frac{\partial}{\partial \tilde{\tau}_2}$. Note that $[\mathcal{L}_1, \tilde{\mathcal{L}}_2] = x_3 \mathcal{L}_0 \neq 0$.

Theorem 7. *The function $\wp = \wp(u; g_2, g_3)$, where $u = \tau_1$ and $g_k = g_k(\tau_0, \tau_2)$, satisfies the heat equation*

$$\frac{\partial}{\partial \tilde{\tau}_2} \wp = \frac{2}{3} \frac{\partial^2}{\partial \tau_1^2} \wp$$

in nonholonomic frame $(\mathcal{L}_0, \mathcal{L}_1, \tilde{\mathcal{L}}_2)$.

See in [8] the derivation of heat equations in nonholonomic frames in the case $g > 1$.

The system $\hat{S}_2 + \alpha x_2 \hat{S}_0$ on \mathbb{C}^3 , where α is a parameter (see Corollary 4), corresponds to the dynamical system

$$\begin{aligned} x_2' &= \frac{2}{3} x_4 + 2(\alpha - 1)x_2^2; \\ x_4' &= 3x_6 + 2(2\alpha + 1)x_2 x_4; \\ x_6' &= 6(\alpha + 1)x_2 x_6. \end{aligned}$$

This system leads to the differential equation in $U = \frac{1}{k}x_2$:

$$U''' - 2k(7\alpha + 2)UU'' - 2k(4\alpha - 1)(U')^2 + 24k^2(3\alpha^2 + \alpha - 1)U^2U' - 24k^3(\alpha + 1)(\alpha - 1)(2\alpha + 1)U^4 = 0.$$

For $\alpha = -2$, $k = -1/12$ we obtain the equation

$$U''' - 2UU'' + 3(U')^2 - \frac{1}{8}(6U' - U^2)^2 = 0$$

with solution (see [13], Corollary 8.4, and [1], formula (33))

$$U(t) = -2 \left(\frac{1}{t - a_1} + \frac{1}{t - a_2} + \frac{1}{t - a_3} \right).$$

3. POLYNOMIAL DYNAMICAL SYSTEMS ON \mathbb{C}^6

Consider the bundle

$$\pi: U_2 \rightarrow B_2$$

of Jacobian varieties of non-singular curves of genus 2

$$V_\lambda = \{(x, y) \in \mathbb{C}^2 : y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\}$$

with base

$$B_2 = \{(\lambda_4, \lambda_6, \lambda_8, \lambda_{10}) \in \mathbb{C}^4 : \Delta(\lambda) \neq 0\},$$

where the discriminant $\Delta(\lambda)$ is described in Appendix B. The fiber over the point $\lambda \in B_2$ is the Jacobi variety $J_\lambda = \mathbb{C}^2/\Gamma_\lambda$ of the curve V_λ , where Γ_λ is a lattice generated by period matrices $2\omega, 2\omega'$ of holomorphic differentials (see Appendix B).

Denote by \mathcal{F}_2 the field of C^∞ -functions on U_2 , whose restrictions to fibers J_λ are hyperelliptic functions of genus 2.

For any point $\lambda \in B_2$ a hyperelliptic sigma-function $\sigma(u; \lambda) = \sigma(u_1, u_3; \lambda)$ of genus 2 is defined uniquely (see Appendix B). It determines the functions

$$\zeta_k(u_1, u_3; \lambda) = \frac{\partial}{\partial u_k} \ln \sigma(u_1, u_3; \lambda), \quad k = 1, 3, \quad (3)$$

and for $i + j \geq 2$ the functions

$$\wp_{i,3j}(u_1, u_3; \lambda) = -\frac{\partial^i}{\partial u_1^i} \frac{\partial^j}{\partial u_3^j} \ln \sigma(u_1, u_3; \lambda). \quad (4)$$

Thus,

$$\frac{\partial}{\partial u_1} \wp_{i,3j} = \wp_{i+1,3j} \quad \text{and} \quad \frac{\partial}{\partial u_3} \wp_{i,3j} = \wp_{i,3(j+1)}.$$

The functions $\wp_{i,3j}(u; \lambda) = \wp_{i,3j}(u_1, u_3; \lambda)$ are Abelian functions in u_1 and u_3 . The function $\sigma(u; \lambda)$ is odd in $u = (u_1, u_3)$, therefore $\wp_{i,3j}(-u; \lambda) = (-1)^{i+j} \wp_{i,3j}(u; \lambda)$.

Please note: to control the grading, we have altered the notation used in our works [5, 6, 11, 12] on the theory of sigma-functions. In the notation of these works we have

$$\wp_{i,3j}(u_1, u_3; \lambda) = \wp_{1,\dots,1,3,\dots,3}(u_1, u_3; \lambda),$$

where the subscript in the right has i ones and j threes.

By setting $\deg u_k = -k$ and $\deg \lambda_j = j$, we obtain that $\deg \wp_{i,3j} = i + 3j$. As will be seen further, this allows to build homogeneous polynomial dynamical systems on \mathbb{C}^6 with appropriately graded coordinates.

Denote the \mathcal{F}_2 -module of derivations of the field \mathcal{F}_2 by $\text{Der}(\mathcal{F}_2)$. In [11] a method for constructing the operators \mathcal{L}_i , $i = 0, 2, 4, 6$ using the operators Q_i , $i = 0, 2, 4, 6$, that annihilate the sigma function $\sigma(u_1, u_3; \lambda)$ is given. Operators \mathcal{L}_i , $i = 0, 2, 4, 6$ along with operators $\mathcal{L}_1 = \frac{\partial}{\partial u_1}$, $\mathcal{L}_3 = \frac{\partial}{\partial u_3}$ give the basis of the \mathcal{F}_2 -module $\text{Der}(\mathcal{F}_2)$. In the proof of Theorem 29 we give a detailed derivation of operators \mathcal{L}_i .

Consider the complex linear space \mathbb{C}^6 with graded variables $X = (x_2, x_3, x_4)$, $Z = (z_4, z_5, z_6)$, $\deg x_i = i$, $\deg z_j = j$, and the graded space \mathbb{C}^4 with coordinates $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, $\deg \lambda_k = k$. In [11] a homogeneous polynomial map

$$\pi_2: \mathbb{C}^6 \rightarrow \mathbb{C}^4, \quad \pi_2(X, Z) = \lambda,$$

was introduced, where

$$\pi_2^* \lambda_4 = \hat{\lambda}_4 = \frac{1}{2}(x_4 - 6x_2^2 - 4z_4), \quad (5)$$

$$\pi_2^* \lambda_6 = \hat{\lambda}_6 = -\frac{1}{4}(2x_2(x_4 + 4z_4) - 8x_2^3 - x_3^2 - 2z_6), \quad (6)$$

$$\pi_2^* \lambda_8 = \hat{\lambda}_8 = -\frac{1}{2}(x_2z_6 + x_4z_4 - 8x_2^2z_4 - 2z_4^2 - x_3z_5), \quad (7)$$

$$\pi_2^* \lambda_{10} = \hat{\lambda}_{10} = \frac{1}{4}(8x_2z_4^2 - 2z_4z_6 + z_5^2). \quad (8)$$

Consider the space \mathbb{C}^7 with graded coordinates $(x_2, x_4, z_4, x_6, z_6, w_8, z_{10})$. From formulas (5)–(8) it follows that the map π_2 decomposes into

$$\pi_2 = \pi_2' \pi_2'' : \mathbb{C}^6 \rightarrow \mathbb{C}^7 \rightarrow \mathbb{C}^4,$$

where $\pi_2''(x_2, x_3, x_4, z_4, z_5, z_6) = (x_2, x_4, z_4, x_3^2, z_6, x_3z_5, z_5^2)$ and

$$(\pi_2')^* \lambda_4 = \frac{1}{2}(x_4 - 6x_2^2 - 4z_4),$$

$$(\pi_2')^* \lambda_6 = -\frac{1}{4}(2x_2(x_4 + 4z_4) - 8x_2^3 - x_6 - 2z_6),$$

$$(\pi_2')^* \lambda_8 = -\frac{1}{2}(x_2z_6 + x_4z_4 - 8x_2^2z_4 - 2z_4^2 - w_8),$$

$$(\pi_2')^* \lambda_{10} = \frac{1}{4}(8x_2z_4^2 - 2z_4z_6 + z_{10}).$$

The image of π'_2 is the hypersurface $C^6 \subset \mathbb{C}^7$ determined by the equation $w_8^2 = x_6 z_{10}$.

The Jacobi matrix $J(\pi_2)$ of π_2 is a (6×4) -matrix with polynomial coefficients. The point $(X, Z) \in \mathbb{C}^6$ is singular if and only if all 15 polynomials, equal to (4×4) -minors of the matrix $J(\pi_2)$, are equal to zero. A direct check shows that if $(X, Z) \in \mathbb{C}^6$ is singular, then $\pi_2(X, Z) \in \Delta$.

Set

$$\tilde{\Delta} = \{(X, Z) \in \mathbb{C}^6 \mid \pi_2(X, Z) \in \Delta\} \quad \text{and} \quad \tilde{W}_2 = \mathbb{C}^6 \setminus \tilde{\Delta}.$$

Thus, the regular algebraic bundle

$$\pi_2: \tilde{W}_2 \rightarrow B_2$$

is defined. It's fibers are non-singular two-dimensional complex algebraic submanifolds in \mathbb{C}^6 .

On the space \tilde{W}_2 there is the involution $\tau(x_2, x_3, x_4, z_4, z_5, z_6) = (x_2, -x_3, x_4, z_4, -z_5, z_6)$.

Set

$$\tilde{U}_2 = \{(u_1, u_3; \lambda) \in U_2 : \sigma(u_1, u_3; \lambda) \neq 0\}.$$

On the space \tilde{U}_2 there is the involution $\tilde{\tau}(u_1, u_3; \lambda) = (-u_1, -u_3; \lambda)$. The map $\tilde{\pi}: \tilde{U}_2 \rightarrow B_2$ is a bundle with the Jacobi variety of the curve V_λ as fiber over point $\lambda \in B_2$. The involution $\tilde{\tau}$ is fiberwise. The quotient space $\tilde{U}_2/\tilde{\tau}$ is the space of a bundle over B_2 with fiber over point $\lambda \in B_2$ the Kummer variety of the curve V_λ (see [4, 17]).

Set $\wp_2 = \wp_{2,0}(u_1, u_3; \lambda)$ and $\wp_4 = \wp_{1,3}(u_1, u_3; \lambda)$. In [11] the following uniformization theorem was proved:

Theorem 8. *The map*

$$\varphi: \tilde{U}_2 \rightarrow \tilde{W}_2, \quad \varphi(u_1, u_3; \lambda) = (X, Z),$$

where $x_2 = \wp_2$, $x_3 = \wp'_2$, $x_4 = \wp''_2$, $z_4 = \wp_4$, $z_5 = \wp'_4$, $z_6 = \wp''_4$ and $f' = \frac{\partial}{\partial u_1} f$, determines a fibered and equivariant with respect to $\tilde{\tau}$ and τ diffeomorphism of bundles

$$\begin{array}{ccc} \tilde{U}_2 & \longrightarrow & \tilde{W}_2 \\ \downarrow & & \downarrow \\ B_2 & \xlongequal{\quad} & B_2 \end{array}.$$

Let us introduce the derivations \mathcal{L}_k , $k = 0, 1, 2, 3, 4, 6$, of the ring of polynomials on the space \mathbb{C}^6 with coordinates (X, Z) such, that

$$L_k \varphi^* P = \varphi^* \mathcal{L}_k P$$

for any polynomial $P = P(X, Z)$.

The homomorphism φ^* is a ring homomorphism, and the operators \mathcal{L}_k are derivations. Thus to construct the operators \mathcal{L}_k it is sufficient to take only the multiplicative generators of the ring $\mathbb{C}[X, Z]$ as polynomials P .

Let us show that the fields \mathcal{L}_k determine graded one-dimensional dynamical systems on \mathbb{C}^6 . The field \mathcal{L}_0 is the Euler field. Thus the system S_0 has the form

$$\begin{aligned} \frac{\partial}{\partial \tau_0} x_k &= k x_k, \quad k = 2, 3, 4, \\ \frac{\partial}{\partial \tau_0} z_l &= k z_l, \quad l = 4, 5, 6. \end{aligned}$$

Set $L_1 f = f'$. Let us introduce the polynomials

$$P_5 = 4(3x_2 x_3 + z_5) \quad \text{and} \quad P_7 = 4(2x_2 z_5 + x_3 z_4). \quad (9)$$

Since $\mathcal{L}_1 = \partial_{u_1}$, using the results of [11, 12] we obtain:

Theorem 9. *System S_1 has the form*

$$\begin{aligned} x'_2 &= x_3, & x'_3 &= x_4, & x'_4 &= P_5, \\ z'_4 &= z_5, & z'_5 &= z_6, & z'_6 &= P_7. \end{aligned}$$

Let us introduce the polynomials

$$G_6 = \frac{1}{2}(6x_2z_4 - z_6). \quad (10)$$

We have

$$G'_6 = 3(x_3z_4 + x_2z_5) - \frac{1}{2}P_7 = x_3z_4 - x_2z_5, \quad (11)$$

$$G''_6 = x_4z_4 - x_2z_6, \quad (12)$$

$$G'''_6 = 8x_2(x_3z_4 - x_2z_5) + (x_4z_5 - x_3z_6) + 4z_4z_5. \quad (13)$$

In the theory of hyperelliptic functions $\wp_{i,3j}$, $i + j \geq 2$, there is a fundamental relation (see [3, 5, 6, 12])

$$\wp_{0,6} = 3\wp_{2,0}\wp_{1,3} - \frac{1}{2}\wp_{3,3}.$$

Thus, $\wp_{0,6} = \varphi^*G_6$. Therefore $\varphi^*\mathcal{L}_kG_6 = L_k\wp_{0,6}$ for all $k = 0, 1, 2, 3, 4, 6$. Since the fields $L_1 = \frac{\partial}{\partial u_1}$ and $L_3 = \frac{\partial}{\partial u_3}$ commute, to define the system S_3 (using Theorem 9) it is sufficient to specify the values of \mathcal{L}_3 on the generators x_2 and z_4 .

Set $L_3f = \dot{f}$.

Theorem 10. *The system S_3 , given system S_1 , is fully determined by the equations*

$$\dot{x}_2 = z_5, \quad \dot{z}_4 = x_3z_4 - x_2z_5.$$

Corollary 11. *The formula holds*

$$\dot{G}_6 = 3(z_4z_5 + x_2G'_6) - \frac{1}{2}G'''_6 = z_4z_5 - x_2(x_3z_4 - x_2z_5) - \frac{1}{2}(x_4z_5 - x_3z_6). \quad (14)$$

Corollary 12. *For any i, j , $i + j \geq 2$ there exist polynomials $\widehat{\wp}_{i,3j}(X, Z)$, such that*

$$\varphi^*\widehat{\wp}_{i,3j}(X, Z) = \wp_{i,3j}(u_1, u_3; \lambda).$$

Using this corollary and the description of commutation relations in the Lie algebra $\text{Der}(\mathcal{F}_2)$ (see Appendix B, Theorem 32), we obtain:

Theorem 13. *There is a polynomial Lie algebra with generators \mathcal{L}_k , $k = 0, 1, 2, 3, 4, 6$, and commutation relations over the ring of polynomials $\mathbb{C}[X, Z]$:*

$$\begin{aligned} [\mathcal{L}_0, \mathcal{L}_k] &= k\mathcal{L}_k, & k &= 1, 2, 3, 4, 6; \\ [\mathcal{L}_1, \mathcal{L}_2] &= x_2\mathcal{L}_1 - \mathcal{L}_3; & [\mathcal{L}_1, \mathcal{L}_3] &= 0; & [\mathcal{L}_1, \mathcal{L}_4] &= z_4\mathcal{L}_1 + x_2\mathcal{L}_3; & [\mathcal{L}_1, \mathcal{L}_6] &= z_4\mathcal{L}_3; \\ [\mathcal{L}_2, \mathcal{L}_4] &= 2\mathcal{L}_6 + \frac{1}{2}x_3\mathcal{L}_3 - \frac{8}{5}\widehat{\lambda}_4\mathcal{L}_2 - \frac{1}{2}z_5\mathcal{L}_1 + \frac{8}{5}\widehat{\lambda}_6\mathcal{L}_0; & [\mathcal{L}_2, \mathcal{L}_3] &= -\left(z_4 + \frac{4}{5}\widehat{\lambda}_4\right)\mathcal{L}_1; \\ [\mathcal{L}_2, \mathcal{L}_6] &= -\frac{4}{5}\widehat{\lambda}_4\mathcal{L}_4 + \frac{1}{2}z_5\mathcal{L}_3 - \frac{1}{2}G'_6\mathcal{L}_1 + \frac{4}{5}\widehat{\lambda}_8\mathcal{L}_0; \\ [\mathcal{L}_3, \mathcal{L}_4] &= (z_4 - \widehat{\lambda}_4)\mathcal{L}_3 + \left(G_6 + \frac{6}{5}\widehat{\lambda}_6\right)\mathcal{L}_1; & [\mathcal{L}_3, \mathcal{L}_6] &= G_6\mathcal{L}_3 + \frac{3}{5}\widehat{\lambda}_8\mathcal{L}_1; \\ [\mathcal{L}_4, \mathcal{L}_6] &= 2\widehat{\lambda}_4\mathcal{L}_6 - \frac{6}{5}\widehat{\lambda}_6\mathcal{L}_4 + \frac{1}{2}G'_6\mathcal{L}_3 + \frac{6}{5}\widehat{\lambda}_8\mathcal{L}_2 - \frac{1}{2}\dot{G}_6\mathcal{L}_1 - 2\widehat{\lambda}_{10}\mathcal{L}_0. \end{aligned}$$

Here the polynomials $\widehat{\lambda}_k$, $k = 4, 6, 8, 10$, are defined by formulas (5)–(8), and the polynomials G_6 , G'_6 , \dot{G}_6 by formulas (10), (11), and (14).

Proof. The commutation relations for the operators \mathcal{L}_k follow from the commutation relations for the operators L_k (see Appendix B) and the description of systems S_1 and S_3 obtained above. The actions of the operators \mathcal{L}_k on the generators of the polynomial ring are described by dynamical systems S_k , $k = 0, 1, 2, 3, 4, 6$. Thus, to finish the proof it is sufficient to describe the action of the operators $\mathcal{L}_2, \mathcal{L}_4$, and \mathcal{L}_6 on the generators x_2 and z_4 .

Let us introduce the polynomials

$$\begin{aligned} P_4 &= \frac{1}{2}x_4 - x_2^2 + 2z_4 + \frac{3}{5}\hat{\lambda}_4 = \frac{1}{5}(4x_4 - 14x_2^2 + 4z_4), \\ F_6 &= z_6 - 2x_2z_4 + \frac{2}{5}\hat{\lambda}_6 = \frac{1}{10}(12z_6 - 28x_2z_4 + 8x_2^3 - 2x_2x_4 + x_3^2), \\ P_8 &= \frac{1}{2}G_6'' - z_4^2 + \frac{1}{5}\hat{\lambda}_8 = \frac{1}{10}(-6x_2z_6 + 4x_4z_4 - 8z_4^2 + 8x_2^2z_4 + x_3z_5). \end{aligned}$$

Using Theorem 29 (see Appendix B) in terms of this polynomials we obtain:

$$\mathcal{L}_0x_2 = 2x_2, \quad \mathcal{L}_1x_2 = x_3, \quad \mathcal{L}_2x_2 = P_4, \quad \mathcal{L}_3x_2 = z_5, \quad \mathcal{L}_4x_2 = F_6, \quad \mathcal{L}_6x_2 = P_8.$$

□

We also introduce polynomials

$$\begin{aligned} P_6 &= \frac{2}{5}(6x_2^3 - x_2x_4 + 9x_2z_4), \\ F_8 &= \frac{1}{10}(-24x_2^4 + 6x_2^2x_4 - 76x_2^2z_4 - 3x_2x_3^2 - 6x_2z_6 - 5x_3z_5 + 20x_4z_4 - 40z_4^2), \\ F_{10} &= \frac{1}{20}(-16x_2x_3z_5 + 10x_3^2z_4 - 4x_2x_4z_4 - 20z_5^2 + 16x_2^2z_6 - 152x_2z_4^2 - 48x_2^3z_4 + 40z_4z_6). \end{aligned}$$

In the following theorem we will need polynomials with two indices $P_{4,1}, \dots, F_{10,2}$. By context it will be clear how to get these polynomials from polynomials with one index P_4, \dots, F_{10} using commutation formulas for the fields $\mathcal{L}_0, \dots, \mathcal{L}_6$ (see Theorem 13).

Example. By definition

$$P_{4,1} = \mathcal{L}_2x_3 = \mathcal{L}_2\mathcal{L}_1x_2.$$

We have $\mathcal{L}_2\mathcal{L}_1 = \mathcal{L}_1\mathcal{L}_2 - x_2\mathcal{L}_1 + \mathcal{L}_3$. Therefore

$$P_{4,1} = P_4' - x_2x_3 + z_5 = 3x_2x_3 + 5z_5.$$

In terms of polynomials that were introduced above we have:

Theorem 14. *The fields $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_6$ are defined by the matrix*

$$\mathcal{T}_2 = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 & 4z_4 & 5z_5 & 6z_6 \\ x_3 & x_4 & P_5 & z_5 & z_6 & P_7 \\ P_4 & P_{4,1} & P_{4,2} & P_6 & P_{6,1} & P_{6,2} \\ z_5 & z_6 & P_7 & G_6' & G_6'' & G_6''' \\ F_6 & F_{6,1} & F_{6,2} & F_8 & F_{8,1} & F_{8,2} \\ P_8 & P_{8,1} & P_{8,2} & F_{10} & F_{10,1} & F_{10,2} \end{pmatrix}.$$

Corollary 15. *The polynomial vector fields*

$$\begin{aligned} \mathcal{L}_1 &= x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + P_5 \frac{\partial}{\partial x_4} + z_5 \frac{\partial}{\partial z_4} + z_6 \frac{\partial}{\partial z_5} + P_7 \frac{\partial}{\partial z_6}, \\ \mathcal{L}_3 &= z_5 \frac{\partial}{\partial x_2} + z_6 \frac{\partial}{\partial x_3} + P_7 \frac{\partial}{\partial x_4} + G_6' \frac{\partial}{\partial z_4} + G_6'' \frac{\partial}{\partial z_5} + G_6''' \frac{\partial}{\partial z_6} \end{aligned}$$

commute and annihilate the polynomials $\hat{\lambda}_k$, $k = 4, 6, 8, 10$.

Consider the space \mathbb{C}^6 with coordinates x_k , $k = 2, \dots, 5$, where $x_5 = x_2'''$, $\hat{\lambda}_{2q}$, $q = 2, 3$.

Corollary 16. *There is the bundle $\pi : \mathbb{C}^6 \rightarrow \mathcal{B} \subset \mathbb{C}^4$, $\pi(x_k, \lambda_{2q}) = (\hat{\lambda}_4, \hat{\lambda}_6, \hat{\lambda}_8, \hat{\lambda}_{10})$, where*

$$\begin{aligned}\hat{\lambda}_8 &= -\frac{1}{16}x_4^2 + \frac{1}{8}x_3x_5 - \frac{5}{4}x_2x_3^2 + \frac{5}{4}x_2^4 + \frac{1}{2}\hat{\lambda}_4x_2^2 - \hat{\lambda}_6x_2 + \frac{1}{4}\hat{\lambda}_4^2, \\ \hat{\lambda}_{10} &= \frac{1}{64}x_5^2 - \frac{3}{8}x_2x_3x_5 - \frac{1}{8}x_2x_4^2 + \frac{1}{16}x_3^2x_4 + \frac{5}{4}x_2^3x_4 + \frac{15}{8}x_2^2x_3^2 - 3x_2^5 - \\ &\quad - \frac{1}{8}\hat{\lambda}_4(x_3^2 - 2x_2x_4 + 8x_2^3) - \frac{1}{4}\hat{\lambda}_6(x_4 - 6x_2^2) + \frac{1}{2}\hat{\lambda}_4\hat{\lambda}_6.\end{aligned}$$

Along the fiber over a point $\hat{\lambda} \in \mathcal{B}$ there acts the operator

$$L_1 = x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} + x_5 \frac{\partial}{\partial x_4} + (10x_3^2 + 20x_2x_4 - 40x_2^3 - 8\hat{\lambda}_4x_2 + 8\hat{\lambda}_6) \frac{\partial}{\partial x_5}.$$

Examples.

$$4(\mathcal{L}_3 + 3x_2\mathcal{L}_1)x_2 = P_5, \quad 4(2x_2\mathcal{L}_3 + z_4\mathcal{L}_1)x_2 = P_7.$$

Note that all matrix elements $(\mathcal{T}_2)_{i,j}$, $1 \leq i, j \leq 6$, of the matrix \mathcal{T}_2 are homogeneous polynomials in (X, Z) , with grading equal to $i + j$.

Corollary 17. *The polynomial $\det \mathcal{T}_2$ in (X, Z) is homogeneous of degree 40.*

Corollary 18. *The polynomial vector fields $\mathcal{L}_0, \mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_6$ on \mathbb{C}^6 under the map $\pi_2'' : \mathbb{C}^6 \rightarrow \mathbb{C}^7$ determine polynomial vector fields $\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_2, \hat{\mathcal{L}}_4, \hat{\mathcal{L}}_6$ on \mathbb{C}^7 such that*

$$\mathcal{L}_k(\pi_2'')^*P = (\pi_2'')^*\hat{\mathcal{L}}_kP, \quad k = 0, 2, 4, 6.$$

Example. We have

$$\mathcal{L}_2x_3^2 = 2x_3P_{4,1} = 6x_2x_3^2 + 10x_3z_5.$$

We obtain

$$\hat{\mathcal{L}}_2x_6 = 6x_2x_6 + 10w_8.$$

4. DIFFERENTIAL EQUATIONS ON THE GENERATORS OF THE DIFFERENTIAL FIELD

The KdV-hierarchy is an infinite system of differential equations

$$U_{t_k} = \chi_k U, \quad k = 1, 2, \dots,$$

on the function $U = U(t_1, t_2, \dots)$. It is determined by the recursion

$$\chi_{k+1}U = \mathcal{R}\chi_kU$$

with initial conditions $\chi_1 = \frac{\partial}{\partial t_1}$, where \mathcal{R} is the Lenard operator

$$\mathcal{R} = \frac{1}{4} \frac{\partial^2}{\partial t_1^2} - U - \frac{1}{2}U_{t_1} \left(\frac{\partial}{\partial t_1} \right)^{-1}.$$

Set $t_1 = u_1$, $t_2 = u_3$. For brevity we use the notation $U_{t_1} = U'$ and $U_{t_3} = \dot{U}$.

Consider the function $x_2(u_1, u_3) = \wp_{2,0}(u_1, u_3)$. Set $U = 2x_2$. Note that for such a function U the Lenard operator \mathcal{R} is homogeneous of degree 2 in our grading. According to Theorem 9 we have

$$x_2''' = x_4' = P_5 = 4(3x_2x_3 + z_5).$$

Since $x_3 = x_2'$ and $z_5 = \dot{x}_2$, we obtain the equation

$$U''' = 6UU' + 4\dot{U}.$$

Theorem 19 (see [5, 6, 12]). *The function $U = 2\wp_{2,0}(u_1, u_3; \lambda)$ satisfies the KdV hierarchy.*

Proof. We have

$$\dot{U} = \chi_2 U = \mathcal{R}\chi_1 U = \mathcal{R}U' = \frac{1}{4}(U''' - 6UU').$$

Further, as $\dot{U} = 2\dot{x}_2 = 2z_4'$, so

$$\mathcal{R}\chi_2 U = 2\mathcal{R}z_4' = \frac{1}{2}z_4''' - (2x_2)(2z_4') - 2x_2'z_4.$$

Hence,

$$2\mathcal{R}\chi_2 U = z_6' - 8x_2z_5 - 4x_3z_4.$$

Since $z_6' = P_7 = 4(2x_2z_5 + x_3z_4)$ (see formula (9)), we obtain that $\mathcal{R}\chi_2 U = 0$. Thus, the function $U = 2\wp_{2,0}(u_1, u_3; \lambda)$ satisfies the KdV equation $U_{t_2} = \chi_2 U$ and the stationary timewise for $t_3 = u_5$ equation $\mathcal{R}\chi_2 U = 0$ of the KdV hierarchy. \square

Using the formula $\dot{z}_4 = 3(x_2z_4)' - \frac{1}{2}z_4'''$ (see Theorem 14), we obtain:

Theorem 20. *The functions $V = 2\wp_{1,3}(u_1, u_3; \lambda)$ and $U = 2\wp_{2,0}(u_1, u_3; \lambda)$ give a solution of the system of equations*

$$\begin{aligned} V''' &= 3(UV)' - 2\dot{V}, \\ U''' &= 3(U^2)' + 4\dot{U}, \\ V' &= \dot{U}. \end{aligned} \tag{15}$$

The following results give for genus $g = 2$ an analogue of corollary 6.

Theorem 21. *The function $U = 2\wp_{2,0}(u_1, u_3; \lambda)$ is a solution of the non-linear differential equation of order six*

$$U'U'''''' - U''U'''' - 10UU'U''' + 10(UU'' - 3(U')^2)U''' + 60U(U')^3 = 0. \tag{16}$$

Proof. By the polynomial dynamical system S_1 , determined by \mathcal{L}_1 , we have:

$$x_2''' = P_5 = 6(x_2^2)' + 4z_5.$$

Consequently,

$$x_2'''' = 6(x_2^2)'' + 4z_6.$$

From equations (5) and (6) we have

$$\begin{aligned} 4z_4 &= x_2'' - 6x_2^2 - 2\hat{\lambda}_4, \\ 2z_6 &= 4x_2(x_2'' - 3x_2^2 - \hat{\lambda}_4) - 8x_2^3 - (x_2')^2 + 4\hat{\lambda}_6. \end{aligned}$$

Thus, we obtain the equation

$$x_2'''' = 20x_2x_2'' + 10(x_2')^2 - 40x_2^3 - 8\hat{\lambda}_4x_2 + 8\hat{\lambda}_6.$$

Set $U = 2x_2$ and pass again to the coordinate system u_1, u_3, λ . Further the prime denotes the derivative with respect to u_1 . Then

$$U'''' = 10UU'' + 5(U')^2 - 10U^3 - 8\lambda_4U + 16\lambda_6.$$

Excluding the parameters λ_4, λ_6 , and differentiating this equation twice we get

$$\begin{aligned} U'''''' &= 10UU''' + 20U'U'' - 30U^2U' - 8\lambda_4U', \\ U'''''' &= 10UU'''' + 30U'U''' + 20(U'')^2 - 30U^2U'' - 8\lambda_4U'' - 60U(U')^2. \end{aligned}$$

Excluding λ_4 from the last two equations, we obtain

$$U'U'''''' - U''U'''' - 10UU'U''' + 10(UU'' - 3(U')^2)U''' + 60U(U')^3 = 0.$$

□

Directly from the proof of this theorem we obtain

Corollary 22. 1. A solution $U(u_1)$ of the fourth-order differential equation with constant a and b

$$U'''' - 10UU'' - 5(U')^2 + 10U^3 + aU + b = 0. \quad (17)$$

is the function $U(u_1) = 2\wp_{2,0}(u_1, u_3; \lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, where $U' = \frac{\partial}{\partial u_1} U$ and $\lambda_4 = \frac{1}{8}a$, $\lambda_6 = -\frac{1}{16}b$.

2. A solution $U(u_1)$ of the fifth-order differential equation with constant a

$$U''''' - 10UU''' - 20U'U'' + 30U^2U' + aU' = 0 \quad (18)$$

is the function $U(u_1) = 2\wp_{2,0}(u_1, u_3; \lambda_4, \lambda_6, \lambda_8, \lambda_{10})$, where $U' = \frac{\partial}{\partial u_1} U$ and $\lambda_4 = \frac{1}{8}a$.

Theorem 23. The function $U = U(u_1) = 2\wp_{2,0}(u_1, u_3; \lambda)$ satisfies the following differential equations:

$$\begin{aligned} U'U''' - \frac{1}{2}(U'')^2 - 5U(U')^2 + \frac{5}{2}U^4 + 4\lambda_4U^2 - 16\lambda_6U + 8\lambda_4^2 - 32\lambda_8 &= 0, \\ (U''')^2 - 10U(U'')^2 + 2(U')^2U'' + 4(5U^3 + 4\lambda_4U - 8\lambda_6)U'' - 2(15U^2 + 4\lambda_4)(U')^2 + B(U) &= 0, \end{aligned}$$

where

$$B(U) = 6U^5 + 16\lambda_4U^3 - 96\lambda_6U^2 + 96(\lambda_4^2 - 4\lambda_8)U + 128\lambda_4\lambda_6 - 256\lambda_{10}.$$

Corollary 24. The function $U = U(u_1) = 2\wp_{2,0}(u_1, u_3; \lambda)$ satisfies the equation

$$(U'')^4 - 20U(U')^2(U'')^2 - A_1(U)(U'')^2 + 8(U')^4U'' + A_2(U)(U')^2U'' - A_3(U)(U')^4 - A_4(U)(U')^2 + A_5(U) = 0,$$

where

$$\begin{aligned} A_1(U) &= 10U^4 + 16\lambda_4U^2 - 64\lambda_6U + 32\lambda_4^2 - 128\lambda_8, \\ A_2(U) &= 80U^3 + 64\lambda_4U - 128\lambda_6, \\ A_3(U) &= 20U^2 + 32\lambda_4, \\ A_4(U) &= 76U^5 + 96\lambda_4U^3 - 256\lambda_6U^2 - 64(\lambda_4^2 - 4\lambda_8)U - 512\lambda_4\lambda_6 + 1024\lambda_{10}, \\ A_5(U) &= (5U^4 + 8\lambda_4U^2 - 32\lambda_6U + 16\lambda_4^2 - 64\lambda_8)^2. \end{aligned}$$

Set $\partial_1 f = \frac{\partial}{\partial u_1} f = f'$.

Consider the Schrödinger operator $S = -\partial_1^2 + U$. In [14] the asymptotic expansion of the resolvent of S was obtained in the form

$$R(x; \zeta) = \sum_{l=0}^{\infty} R_l(U) \frac{1}{\zeta^{l+\frac{1}{2}}}$$

for $\zeta \rightarrow \infty$. The coefficients of this expansion are differential polynomials in U by ∂_1 . They are related by the recursion (see the action of the Lenard operator)

$$R'_{l+1} = \frac{1}{4}R''_l - UR'_l - \frac{1}{2}U'R_l$$

with initial condition $R_0 = \frac{1}{2}$.

In [14] an algorithm to calculate the polynomials R_l was described.

We have

$$\begin{aligned} R_1 &= -\frac{1}{4}U, \\ R_2 &= \frac{1}{16}(3U^2 - U''), \\ R_3 &= -\frac{1}{64}(10U^3 - 10UU'' - 5(U')^2 + U^{IV}), \\ R_4 &= \frac{1}{256}(35U^4 - 70U(U')^2 - 70U^2U'' + 21(U'')^2 + 28U'U''' + 14UU^{IV} - U^{VI}). \end{aligned}$$

S. P. Novikov's equation (the higher stationary KdV equation) is the ordinary differential equation of order $2n$

$$\sum_{k=0}^{n+1} c_k R_k(U) = 0$$

on the function $U(u_1)$. Here $c_{n+1} = 1$ and c_k are arbitrary constants. This equation has n independent integrals I_l , $l = 1, \dots, n$.

Thus we obtain that the equation (17) is the equation

$$-64R_3 - 4aR_1 + 2bR_0 = 0.$$

The integrals I_1 and I_2 of this equation are differential polynomials in U that determine the parameters λ_8 and λ_{10} of the curve.

In [18] in the section "Transcendents determined by nonlinear fourth-order equations" much attention is given to the non-autonomous differential equation

$$Y'''' + 10YY'' + 5(Y')^2 + 10Y^3 + \alpha Y - z = 0$$

where $Y' = \frac{\partial Y}{\partial z}$. The change of variables $Y = \lambda^{-2}\tilde{Y}$, $z = \lambda\tilde{z}$, where λ is some parameter, brings this equation into

$$\tilde{Y}'''' + 10\tilde{Y}\tilde{Y}'' + 5\tilde{Y}'^2 + 10\tilde{Y}^3 = \lambda^7\tilde{z} - \alpha\lambda^4\tilde{Y}, \quad (19)$$

where the prime denotes the derivative with respect to \tilde{z} . Note that in [18] a misprint was made: the equation (19) is given without the term $-\alpha\lambda^4\tilde{Y}$.

For $\lambda = 0$ equation (19) with $\tilde{Y} = -U$ becomes a special case of our equation (17) for $a = b = 0$, that is an equation corresponding to a KdV equation solution on the Jacobi variety of the curve

$$\{(x, y) \in \mathbb{C}^2 \mid y^2 = x^5 + \lambda_8 x + \lambda_{10}\}.$$

Using formulas (5)–(8), we obtain two integrals of this equation that correspond to parameters λ_8 and λ_{10} .

5. APPENDIX A NECESSARY INFORMATION ON ELLIPTIC FUNCTIONS

For each elliptic curve, with affine part of the form

$$V_g = \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2x - g_3\},$$

the sigma-function $\sigma(u; g_2, g_3)$ is constructed (see [25, 9]). This function is an entire function in $u \in \mathbb{C}$ with parameters $(g_2, g_3) \in \mathbb{C}^2$. It has a series expansion in u over the polynomial ring $\mathbb{Q}[g_2, g_3]$ in the vicinity of 0 (see, for example, [9]). The initial segment of the expansion has the form

$$\sigma(u) = u - \frac{g_2}{2} \frac{u^5}{5!} - 6g_3 \frac{u^7}{7!} - \frac{9g_2^2}{4} \frac{u^9}{9!} - 18g_2g_3 \frac{u^{11}}{11!} + (u^{13}). \quad (20)$$

In [9] a classical recursion is described. It allows to restore all homogeneous polynomials in g_2, g_3 that give the coefficients of the expansion of $\sigma(u; g_2, g_3)$ as a series in u .

We need the following properties of the sigma function:

1. The system of equations holds

$$Q_0\sigma = 0, \quad Q_2\sigma = 0, \quad \text{where} \quad Q_0 = \ell_0 - H_0, \quad Q_2 = \ell_2 - H_2, \quad (21)$$

$$\begin{pmatrix} \ell_0 \\ \ell_2 \end{pmatrix} = T \begin{pmatrix} \frac{\partial}{\partial g_2} \\ \frac{\partial}{\partial g_3} \end{pmatrix}, \quad T = \begin{pmatrix} 4g_2 & 6g_3 \\ 6g_3 & \frac{1}{3}g_2^2 \end{pmatrix}, \quad (22)$$

$$H_0 = u \frac{\partial}{\partial u} - 1, \quad H_2 = \frac{1}{2} \frac{\partial^2}{\partial u^2} + \frac{1}{24} g_2 u^2. \quad (23)$$

Theorem 25 (Uniqueness conditions for the sigma-function).

The entire function $\sigma(u; g_2, g_3)$ is uniquely determined by the conditions:

$$Q_0\sigma = 0, \quad Q_2\sigma = 0,$$

$$\sigma(0; g_2, g_3) = 0, \quad \left(\frac{\partial}{\partial u} \sigma(u; g_2, g_3) \right) \Big|_{u=0} = 1.$$

2. From equation $Q_0\sigma = 0$ it follows that σ is a homogeneous function of degree -1 in u, g_2, g_3 with respect to the grading $\deg u = -1, \deg g_2 = 4, \deg g_3 = 6$.

3. The discriminant of the curve V_g is equal to $\Delta = g_2^3 - 27g_3^2 = \frac{3}{4} \det T$. Set $\mathcal{B} = \{g = (g_2, g_3) \in \mathbb{C}^2 \mid \Delta \neq 0\}$, then the curve V_g , where $g = (g_2, g_3) \in \mathcal{B}$, is non-singular. The fields ℓ_0 and ℓ_2 are tangent to the discriminant manifold $\{(g_2, g_3) \in \mathbb{C}^2, \Delta(g_2, g_3) = 0\}$, since $\ell_0\Delta = 12\Delta, \ell_2\Delta = 0$.

Thus, the fields ℓ_0 and ℓ_2 determine the derivations of the local ring $\mathbb{C}[g_2, g_3]/(\Delta)$.

The function $\sigma(u)$ is quasiperiodic in u with respect to the lattice generated by $(2\omega, 2\omega')$:

$$\sigma(u + 2\omega) = e^{2\eta(u+\omega)} \sigma(u), \quad \sigma(u + 2\omega') = e^{2\eta'(u+\omega')} \sigma(u).$$

The parameters ω, ω', η , and η' are determined by the relations

$$2\omega = \oint_a \frac{dx}{y}, \quad 2\omega' = \oint_b \frac{dx}{y}, \quad 2\eta = - \oint_a x \frac{dx}{y}, \quad 2\eta' = - \oint_b x \frac{dx}{y},$$

where $\frac{dx}{y}$ and $x \frac{dx}{y}$ form the basis of holomorphic differentials on $V_{(g_2, g_3)}$, and a, b are basic cycles on the curve such that the integrals satisfy the Legendre identity $\eta\omega' - \omega\eta' = \frac{\pi i}{2}$.

The function $\zeta(u)$ is determined by the relation $\zeta(u) = \frac{\partial}{\partial u} \ln \sigma(u)$ and is expressed by the series

$$\zeta(u; \omega, \omega') = \frac{1}{u} + \sum_{n^2+m^2 \neq 0} \left(\frac{1}{u - 2m\omega - 2n\omega'} + \frac{1}{2m\omega - 2n\omega'} + \frac{u}{(2m\omega - 2n\omega')^2} \right). \quad (24)$$

Theorem 26 (Weierstrass, see [25]). *There is the following uniformization of the elliptic curve V_g (see equation (2))*

$$\wp'(u)^2 = 4\wp^3(u) - g_2\wp(u) - g_3.$$

6. APPENDIX B

NECESSARY INFORMATION ON SIGMA-FUNCTIONS OF GENUS 2

For each elliptic curve, with affine part of the form

$$V_\lambda = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\},$$

a sigma-function $\sigma(u; \lambda)$ is constructed (see [6]). This function is an entire function in $u = (u_1, u_3)^\top \in \mathbb{C}^2$ with parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})^\top \in \mathbb{C}^4$. It has a series expansion in u over the polynomial ring $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ in the vicinity of 0. The initial segment of the expansion has the form

$$\begin{aligned} \sigma(u; \lambda) = & u_3 - \frac{1}{3} u_1^3 + \frac{1}{6} \lambda_6 u_3^3 - \frac{1}{12} \lambda_4 u_1^4 u_3 - \frac{1}{6} \lambda_6 u_1^3 u_3^2 - \\ & - \frac{1}{6} \lambda_8 u_1^2 u_3^3 - \frac{1}{3} \lambda_{10} u_1 u_3^4 + \left(\frac{1}{60} \lambda_4 \lambda_8 + \frac{1}{120} \lambda_6^2 \right) u_3^5 + (u^7). \end{aligned} \quad (25)$$

Hereinafter, (u^k) denotes the ideal generated by monomials $u_1^i u_3^j$, $i + j = k$.

The sigma-function is an odd function in u , i.e. $\sigma(-u; \lambda) = -\sigma(u; \lambda)$.

Set $\nabla_\lambda = \left(\frac{\partial}{\partial \lambda_4}, \frac{\partial}{\partial \lambda_6}, \frac{\partial}{\partial \lambda_8}, \frac{\partial}{\partial \lambda_{10}} \right)^\top$ and $\partial_{u_1} = \frac{\partial}{\partial u_1}$, $\partial_{u_3} = \frac{\partial}{\partial u_3}$.

We need the following properties of the two-dimensional sigma-function (see details in [6], [9]):

1. The system of equations holds

$$Q_i \sigma = 0, \quad \text{where} \quad Q_i = \ell_i - H_i, \quad i = 0, 2, 4, 6, \quad (26)$$

$$(\ell_0 \ell_2 \ell_4 \ell_6)^\top = T \nabla_\lambda, \quad (27)$$

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix},$$

$$\begin{aligned} H_0 &= u_1 \partial_{u_1} + 3u_3 \partial_{u_3} - 3, \\ H_2 &= \frac{1}{2} \partial_{u_1}^2 - \frac{4}{5} \lambda_4 u_3 \partial_{u_1} + u_1 \partial_{u_3} - \frac{3}{10} \lambda_4 u_1^2 + \frac{1}{10} (15\lambda_8 - 4\lambda_4^2) u_3^2, \\ H_4 &= \partial_{u_1} \partial_{u_3} - \frac{6}{5} \lambda_6 u_3 \partial_{u_1} + \lambda_4 u_3 \partial_{u_3} - \frac{1}{5} \lambda_6 u_1^2 + \lambda_8 u_1 u_3 + \frac{1}{10} (30\lambda_{10} - 6\lambda_6 \lambda_4) u_3^2 - \lambda_4, \\ H_6 &= \frac{1}{2} \partial_{u_3}^2 - \frac{3}{5} \lambda_8 u_3 \partial_{u_1} - \frac{1}{10} \lambda_8 u_1^2 + 2\lambda_{10} u_1 u_3 - \frac{3}{10} \lambda_8 \lambda_4 u_3^2 - \frac{1}{2} \lambda_6. \end{aligned} \quad (28)$$

2. From equation $Q_0 \sigma = 0$ it follows that σ is a homogeneous function of degree -3 in u_1, u_3, λ_j with respect to the grading $\deg u_1 = -1$, $\deg u_3 = -3$, $\deg \lambda_j = j$, and $\deg Q_i = i$, $i = 0, 2, 4, 6$.

3. The discriminant of the hyperelliptic curve V_λ of genus 2 is equal to $\Delta = \frac{16}{5} \det T$. It is a homogeneous polynomial in λ of degree 40. Set $\mathcal{B} = \{\lambda \in \mathbb{C}^4 \mid \Delta(\lambda) \neq 0\}$, then the curve V_λ for $\lambda \in \mathcal{B}$ is non-singular.

We have

$$\ell_0 \Delta = 40\Delta, \quad \ell_2 \Delta = 0, \quad \ell_4 \Delta = 12\lambda_4 \Delta, \quad \ell_6 \Delta = 4\lambda_6 \Delta.$$

Thus, the fields ℓ_0, ℓ_2, ℓ_4 and ℓ_6 are tangent to the manifold $\{\lambda \in \mathbb{C}^4 : \Delta(\lambda) = 0\}$.

4. The parameters matrices ω , ω' , η , and η' are determined by the relations (pages 8–9 in [6])

$$\begin{aligned} 2\omega &= \left(\oint_{a_i} du_j \right)_{i,j=1,3}, & 2\omega' &= \left(\oint_{b_i} du_j \right)_{i,j=1,3}, \\ 2\eta &= - \left(\oint_{a_i} dr_j \right)_{i,j=1,3}, & 2\eta' &= - \left(\oint_{b_i} dr_j \right)_{i,j=1,3}, \end{aligned}$$

where the differential forms

$$du_1 = x \frac{dx}{y}, \quad du_3 = \frac{dx}{y}, \quad dr_1 = x^2 \frac{dx}{y}, \quad dr_3 = (3x^3 + \lambda_4 x) \frac{dx}{y}$$

form the basis of holomorphic differentials on V_λ , and a_i , b_i are basic cycles on the curve, chosen in a way that the Legendre relation holds

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = \frac{\pi i}{2}.$$

The Legendre relation is equivalent to the existence of symmetric matrices τ and \varkappa such that $\omega' = \omega\tau$, $\eta = 2\varkappa\omega$ and $\eta' = 2\varkappa\omega' - \frac{\pi i}{2}(\omega^\top)^{-1}$.

5. The function $\sigma(u; \lambda)$ is quasiperiodic in u with respect to the lattice generated by the (2×4) -matrix of periods $(2\omega, 2\omega')$.

Set $\Omega_1(\mathbf{m}, \mathbf{m}') = \omega\mathbf{m} + \omega'\mathbf{m}'$ and $\Omega_2(\mathbf{m}, \mathbf{m}') = \eta\mathbf{m} + \eta'\mathbf{m}'$, where $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^2$. Then

$$\begin{aligned} \sigma(u + 2\Omega_1(\mathbf{m}, \mathbf{m}'); \lambda) &= \\ &= \exp \left\{ 2\Omega_2^\top(\mathbf{m}, \mathbf{m}')(\mathbf{u} + \Omega_1(\mathbf{m}, \mathbf{m}')) \right\} \exp \left\{ -2\pi(\epsilon^\top \mathbf{m}' + \frac{1}{2} \mathbf{m}^\top \mathbf{m}') \right\} \sigma(u; \lambda). \end{aligned}$$

In particular, for $\mathbf{m}' = 0$ we have

$$\sigma(u + 2\omega\mathbf{m}; \lambda) = \exp \left\{ 2\mathbf{m}^\top \eta^\top (u + \omega\mathbf{m}) \right\} \sigma(u; \lambda).$$

The present work is based on the following results.

Theorem 27 (Uniqueness conditions for the two-dimensional sigma-function).

The entire function $\sigma(u; \lambda)$ is uniquely determined by the system of equations (26) and initial conditions $\sigma(u; 0) = u_3 - \frac{1}{3}u_1^3$.

Set $\sigma(u; \lambda) = u_3 - \frac{1}{3}u_1^3 + \sum_{i,j} p_{ij}(\lambda)u_1^i u_3^j$. In [9] a recursion that allows to recover the polynomials $p_{ij}(\lambda)$ from the initial segment of this expansion is given.

Theorem 28 (see [3, 5, 6, 12]). *The Abelian functions $\wp_{i,3j}$ and parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10})$ are related by the system of equations*

$$\begin{aligned} \wp_{4,0} &= 6\wp_{2,0}^2 + 4\wp_{1,3} + 2\lambda_4, \\ \wp_{3,3} &= 6\wp_{2,0}\wp_{1,3} - 2\wp_{0,6}, \\ \wp_{2,6} &= 2\wp_{2,0}\wp_{0,6} + 4\wp_{1,3}^2 + 2\lambda_4\wp_{1,3}, \\ \wp_{1,9} &= 6\wp_{1,3}\wp_{0,6} + 4\lambda_6\wp_{1,3} - 2\lambda_8\wp_{2,0} - 4\lambda_{10}, \\ \wp_{0,12} &= 6\wp_{0,6}^2 - 12\lambda_{10}\wp_{2,0} + 4\lambda_8\wp_{1,3} + 4\lambda_6\wp_{0,6} + 2\lambda_8\lambda_4. \end{aligned}$$

From this conditions it follows that for any i and j , $i + j \geq 2$, the function $\wp_{i,3j}$ is a differential polynomial in $\wp_{2,0}$ and $\wp_{1,3}$.

Set $L_1 = \partial_{u_1}$, $L_3 = \partial_{u_3}$. We introduce the operators $L_i \in \text{Der } \mathcal{F}_2$, $i = 0, 2, 4, 6$ based on the operators $Q_i = \ell_i - H_i$.

Theorem 29. *The generators of the \mathcal{F}_2 -module $\text{Der } \mathcal{F}_2$ are given by the formulas*

$$\begin{aligned} L_0 &= \ell_0 - u_1 \partial_{u_1} - 3u_3 \partial_{u_3}, \\ L_1 &= \partial_{u_1}, \\ L_2 &= \ell_2 + \left(-\zeta_1 + \frac{4}{5} \lambda_4 u_3 \right) \partial_{u_1} - u_1 \partial_{u_3}, \\ L_3 &= \partial_{u_3}, \\ L_4 &= \ell_4 + \left(-\zeta_3 + \frac{6}{5} \lambda_6 u_3 \right) \partial_{u_1} - (\zeta_1 + \lambda_4 u_3) \partial_{u_3}, \\ L_6 &= \ell_6 + \frac{3}{5} \lambda_8 u_3 \partial_{u_1} - \zeta_3 \partial_{u_3}. \end{aligned} \tag{29}$$

On the differential ring with respect to commuting operators ∂_{u_1} and ∂_{u_3} , generated by $x_2 = \wp_{2,0}(u_1, u_3; \lambda)$ and $z_4 = \wp_{1,3}(u_1, u_3; \lambda)$, the operators L_i , $i = 0, 1, 2, 3, 4, 6$, act as follows:

$$\begin{aligned} L_0 x_2 &= 2x_2; & L_0 z_4 &= 4z_4; \\ L_1 x_2 &= x_3; & L_1 z_4 &= z_5; \\ L_2 x_2 &= \frac{1}{2} x_4 - x_2^2 + 2z_4 + \frac{3}{5} \lambda_4 = P_4; & L_2 z_4 &= \frac{1}{2} z_6 - x_2 z_4 - \frac{4}{5} \lambda_4 x_2 + G_6 = P_6; \\ L_3 x_2 &= z_5; & L_3 z_4 &= \partial_{u_1} G_6; \\ L_4 x_2 &= z_6 - 2x_2 z_4 + \frac{2}{5} \lambda_6 = F_6; & L_4 z_4 &= G_6'' - z_4^2 - x_2 G_6 - \frac{6}{5} \lambda_6 x_2 + \lambda_4 z_4 - \lambda_8 = F_8; \\ L_6 x_2 &= \frac{1}{2} G_6'' - z_4^2 + \frac{1}{5} \lambda_8 = P_8; & L_6 z_4 &= \frac{1}{2} G_6' - z_4 G_6 - \frac{3}{5} \lambda_8 x_2 - 2\lambda_{10} = F_{10}. \end{aligned} \tag{30}$$

Proof. We will use the methods of [11] that allow to obtain the explicit form of operators L_i and describe their action on the ring \mathcal{F}_2 . We note here that this theorem corrects misprints made in [11].

We have $L_1 = \partial_{u_1} \in \text{Der } \mathcal{F}_2$ and $L_3 = \partial_{u_3} \in \text{Der } \mathcal{F}_2$.

Further we use that $[\partial_{u_k}, \ell_q] = 0$, $k = 1, 3$ and $q = 0, 2, 4, 6$.

1). Derivation of the formula for L_0 .

We have $\ell_0 = H_0 = u_1 \partial_{u_1} + 3u_3 \partial_{u_3} - 3$. Therefore

$$\ell_0 \ln \sigma = u_1 \partial_{u_1} \ln \sigma + 3u_3 \partial_{u_3} \ln \sigma - 3. \tag{31}$$

We apply to (31) the operators ∂_{u_1} and ∂_{u_3} . We obtain

$$\ell_0 \zeta_1 = \zeta_1 - u_1 \wp_{2,0} - 3u_3 \wp_{1,3}, \tag{32}$$

$$\ell_0 \zeta_3 = 3\zeta_3 - u_1 \wp_{1,3} - 3u_3 \wp_{0,6}. \tag{33}$$

We apply to (32) the operator ∂_{u_1} to obtain

$$-\ell \wp_{2,0} = -2\wp_{2,0} - u_1 \wp_{3,0} - 3u_3 \wp_{2,3}.$$

Therefore

$$(\ell_0 - u_1 \partial_{u_1} - 3u_3 \partial_{u_3}) \wp_{2,0} = 2\wp_{2,0}.$$

We apply to (33) the operator ∂_{u_1} to obtain

$$-\ell_0 \wp_{1,3} = -\wp_{1,3} - u_1 \wp_{2,3} - 3\wp_{1,3} - 3u_3 \wp_{1,6}.$$

Therefore

$$(\ell_0 - u_1 \partial_{u_1} - 3u_3 \partial_{u_3}) \wp_{1,3} = 4\wp_{1,3}.$$

Thus, we have proved that

$$L_0 = (\ell_0 - u_1 \partial_{u_1} - 3u_3 \partial_{u_3}) \in \text{Der } \mathcal{F}_2$$

and

$$L_0 x_2 = 2x_2, \quad L_0 z_4 = 4z_4.$$

2). Derivation of the formula for L_2 .

We have $\ell_2 = H_2 = \frac{1}{2} \partial_{u_1}^2 - \frac{4}{5} \lambda_4 u_3 \partial_{u_1} + u_1 \partial_{u_3} + w_2$, where $w_2 = w_2(u_1, u_3) = -\frac{3}{10} \lambda_4 u_1^2 + \frac{1}{10} (15\lambda_8 - 4\lambda_4^2) u_3^2$. Therefore

$$\ell_2 \ln \sigma = \frac{1}{2} \frac{\partial_{u_1}^2 \sigma}{\sigma} - \frac{4}{5} \lambda_4 u_3 \partial_{u_1} \ln \sigma + u_1 \partial_{u_3} \ln \sigma + w_2.$$

The formula holds

$$\frac{\partial_{u_1}^2 \sigma}{\sigma} = -\wp_{2,0} + \zeta_1^2.$$

We get

$$\ell_2 \ln \sigma = -\frac{1}{2} \wp_{2,0} + \frac{1}{2} \zeta_1^2 - \frac{4}{5} \lambda_4 u_3 \zeta_1 + u_1 \zeta_3 + w_2. \quad (34)$$

We apply to (34) the operators ∂_{u_1} and ∂_{u_3} . We obtain

$$\begin{aligned} \ell_2 \zeta_1 &= -\frac{1}{2} \wp_{3,0} - \zeta_1 \wp_{2,0} + \frac{4}{5} \lambda_4 u_3 \wp_{2,0} + \zeta_3 - u_1 \wp_{1,3} + \partial_{u_1} w_2, \\ \ell_2 \zeta_3 &= -\frac{1}{2} \wp_{2,3} - \zeta_1 \wp_{1,3} - \frac{4}{5} \lambda_4 \zeta_1 + \frac{4}{5} \lambda_4 u_3 \wp_{1,3} - u_1 \wp_{0,6} + \partial_{u_3} w_2. \end{aligned}$$

Applying again the operator ∂_{u_1} , we obtain

$$\begin{aligned} -\ell_2 \wp_{2,0} &= -\frac{1}{2} \wp_{4,0} + \wp_{2,0}^2 - \zeta_1 \wp_{3,0} + \frac{4}{5} \lambda_4 u_3 \wp_{3,0} - 2\wp_{1,3} - u_1 \wp_{2,3} + \partial_{u_1}^2 w_2, \\ -\ell_2 \wp_{1,3} &= -\frac{1}{2} \wp_{3,3} + \wp_{2,0} \wp_{1,3} - \zeta_1 \wp_{2,3} + \frac{4}{5} \lambda_4 \wp_{2,0} + \frac{4}{5} \lambda_4 u_3 \wp_{2,3} - \wp_{0,6} - u_1 \wp_{1,6} + \partial_{u_1} \partial_{u_3} w_2. \end{aligned}$$

Thus, we have proved that

$$L_2 = (\ell_2 - \zeta_1 \partial_{u_1} - u_1 \partial_{u_3} + \frac{4}{5} \lambda_4 u_3 \partial_{u_1}) \in \text{Der } \mathcal{F}_2.$$

We have $\partial_{u_1}^2 w_2 = -\frac{3}{5}$ and $\partial_{u_1} \partial_{u_3} w_2 = 0$. Therefore

$$\begin{aligned} L_2 x_2 &= \frac{1}{2} x_4 - x_2^2 + 2z_4 + \frac{3}{5} \lambda_4 = P_4, \\ L_2 z_4 &= \frac{1}{2} z_6 - x_2 z_4 - \frac{4}{5} \lambda_4 x_2 + G_6. \end{aligned}$$

Substituting the expressions for λ_4 and G_6 , we obtain the formula

$$L_2 z_4 = P_6.$$

3). Derivation of the formula for L_4 .

We have $\ell_4 = H_4 = \partial_{u_1} \partial_{u_3} - \frac{6}{5} \lambda_6 u_3 \partial_{u_1} + \lambda_4 u_3 \partial_{u_3} + w_4$, where $w_4 = -\frac{1}{5} \lambda_6 u_1^2 + \lambda_8 u_1 u_3 + \frac{1}{10} (30\lambda_{10} - 6\lambda_6 \lambda_4) u_3^2 - \lambda_4$. Therefore

$$\ell_4 \ln \sigma = \frac{\partial_{u_1} \partial_{u_3} \sigma}{\sigma} - \frac{6}{5} \lambda_6 u_3 \partial_{u_1} \ln \sigma + \lambda_4 u_3 \partial_{u_3} \ln \sigma + w_4.$$

The formula holds

$$\frac{\partial_{u_1} \partial_{u_3} \sigma}{\sigma} = -\wp_{1,3} + \zeta_1 \zeta_3.$$

We obtain

$$\ell_4 \ln \sigma = -\wp_{1,3} + \zeta_1 \zeta_3 - \frac{6}{5} \lambda_6 u_3 \zeta_1 + \lambda_4 u_3 \zeta_3 + w_4. \quad (35)$$

We apply to (35) the operators ∂_{u_1} and ∂_{u_3} . We obtain

$$\begin{aligned} \ell_4 \zeta_1 &= -\wp_{2,3} - \wp_{2,0} \zeta_3 - \zeta_1 \wp_{1,3} + \frac{6}{5} \lambda_6 u_3 \wp_{2,0} - \lambda_4 u_3 \wp_{1,3} + \partial_{u_1} w_4, \\ \ell_4 \zeta_3 &= -\wp_{1,6} - \wp_{1,3} \zeta_3 - \zeta_1 \wp_{0,6} - \frac{6}{5} \lambda_6 \zeta_1 + \frac{6}{5} \lambda_6 u_3 \wp_{1,3} + \lambda_4 \zeta_3 - \lambda_4 u_3 \wp_{0,6} + \partial_{u_3} w_4. \end{aligned}$$

By applying again the operator ∂_{u_1} , we obtain

$$\begin{aligned} -\ell_4 \wp_{2,0} &= -\wp_{3,3} - \wp_{3,0} \zeta_3 + \wp_{2,0} \wp_{1,3} + \wp_{2,0} \wp_{1,3} - \zeta_1 \wp_{2,3} + \frac{6}{5} \lambda_6 u_3 \wp_{3,0} - \lambda_4 u_3 \wp_{2,3} + \partial_{u_1}^2 w_4, \\ -\ell_4 \wp_{1,3} &= -\wp_{2,6} - \wp_{2,3} \zeta_3 + \wp_{1,3}^2 + \wp_{2,0} \wp_{0,6} - \\ &\quad - \zeta_1 \wp_{1,6} + \frac{6}{5} \lambda_6 \wp_{2,0} + \frac{6}{5} \lambda_6 u_3 \wp_{2,3} - \lambda_4 \wp_{1,3} - \lambda_4 u_3 \wp_{1,6} + \partial_{u_1} \partial_{u_3} w_4. \end{aligned}$$

Therefore, we have proved that

$$L_4 = (\ell_4 - \zeta_3 \partial_{u_1} - \zeta_1 \partial_{u_3} + \frac{6}{5} \lambda_6 u_3 \partial_{u_1} - \lambda_4 u_3 \partial_{u_3}) \in \text{Der } \mathcal{F}_2.$$

We have $\partial_{u_1}^2 w_4 = -\frac{2}{5} \lambda_6$ and $\partial_{u_1} \partial_{u_3} w_4 = \lambda_8$. Therefore

$$\begin{aligned} L_4 x_2 &= z_6 - 2x_2 z_4 + \frac{2}{5} \lambda_6 = F_6, \\ L_4 z_4 &= G_6'' - z_4^2 - x_2 G_6 - \frac{6}{5} \lambda_6 x_2 + \lambda_4 z_4 - \lambda_8 = F_8. \end{aligned}$$

4). Derivation of the formula for L_6 .

We have $\ell_6 = H_6 = \frac{1}{2} \partial_{u_3}^2 - \frac{3}{5} \lambda_8 u_3 \partial_{u_1} + w_6$, where $w_6 = -\frac{1}{10} \lambda_8 u_1^2 + 2\lambda_{10} u_1 u_3 - \frac{3}{10} \lambda_8 \lambda_4 u_3^2 - \frac{1}{2} \lambda_6$. Therefore

$$\ell_6 \ln \sigma = \frac{1}{2} \frac{\partial_{u_3}^2 \sigma}{\sigma} - \frac{3}{5} \lambda_8 u_3 \partial_{u_1} \ln \sigma + w_6.$$

We obtain

$$\ell_6 \ln \sigma = -\frac{1}{2} \wp_{0,6} + \frac{1}{2} \zeta_3^2 - \frac{3}{5} \lambda_8 u_3 \zeta_1 + w_6. \quad (36)$$

We apply to (36) the operators ∂_{u_1} and ∂_{u_3} . We obtain

$$\begin{aligned} \ell_6 \zeta_1 &= -\frac{1}{2} \wp_{1,6} - \zeta_3 \wp_{1,3} + \frac{3}{5} \lambda_8 u_3 \wp_{2,0} + \partial_{u_1} w_6, \\ \ell_6 \zeta_3 &= -\frac{1}{2} \wp_{0,9} - \zeta_3 \wp_{0,6} - \frac{3}{5} \lambda_8 \zeta_1 + \frac{3}{5} \lambda_8 u_3 \wp_{1,3} + \partial_{u_3} w_6. \end{aligned}$$

By applying again the operator ∂_{u_1} , we obtain

$$\begin{aligned} -\ell_6 \wp_{2,0} &= -\frac{1}{2} \wp_{2,6} + \wp_{1,3}^2 - \zeta_3 \wp_{2,3} + \frac{3}{5} \lambda_8 u_3 \wp_{3,0} + \partial_{u_1}^2 w_6, \\ -\ell_6 \wp_{1,3} &= -\frac{1}{2} \wp_{1,9} + \wp_{1,3} \wp_{0,6} - \zeta_3 \wp_{1,6} + \frac{3}{5} \lambda_8 \wp_{2,0} + \frac{3}{5} \lambda_8 u_3 \wp_{2,3} + \partial_{u_1} \partial_{u_3} w_6. \end{aligned}$$

Therefore, we have proved that

$$L_6 = (\ell_6 - \zeta_3 \partial_{u_3} + \frac{3}{5} \lambda_8 u_3 \partial_{u_1}) \in \text{Der } \mathcal{F}_2.$$

We have $\partial_{u_1}^2 w_6 = -\frac{1}{5} \lambda_8$ and $\partial_{u_1} \partial_{u_3} w_6 = 2\lambda_{10}$. Therefore

$$\begin{aligned} L_6 x_2 &= \frac{1}{2} G_6'' - z_4^2 + \frac{1}{5} \lambda_8 = P_8, \\ L_6 z_4 &= \frac{1}{2} \dot{G}_6' - z_4 G_6 - \frac{3}{5} \lambda_8 x_2 - 2\lambda_{10} = F_{10}. \end{aligned}$$

This ends the proof. □

The description of commutation relations in the differential algebra of Abelian functions of genus 2 was given in [11], see also [12]. We obtain this result directly from Theorem 29 and correct some misprints of [11]. To simplify the calculations we use the following results:

Lemma 30. *The following commutation relations on ℓ_k hold:*

$$\begin{aligned} [\partial_{u_1}, \ell_k] &= 0, \quad k = 0, 2, 4, 6, & [\partial_{u_3}, \ell_k] &= 0, \quad k = 0, 2, 4, 6, \\ [\ell_0, \ell_k] &= k\ell_k, \quad k = 2, 4, 6, & [\ell_2, \ell_4] &= \frac{8}{5}\lambda_6\ell_0 - \frac{8}{5}\lambda_4\ell_2 + 2\ell_6, \\ [\ell_2, \ell_6] &= \frac{4}{5}\lambda_8\ell_0 - \frac{4}{5}\lambda_4\ell_4, & [\ell_4, \ell_6] &= -2\lambda_{10}\ell_0 + \frac{6}{5}\lambda_8\ell_2 - \frac{6}{5}\lambda_6\ell_4 + 2\lambda_4\ell_6. \end{aligned}$$

Proof. This relations follow directly from (27). □

Lemma 31. *The operators L_i , $i = 0, 1, 2, 3, 4, 6$, act on $-\zeta_1$ and $-\zeta_3$ according to the formulas*

$$\begin{aligned} L_0(-\zeta_1) &= -\zeta_1, & L_0(-\zeta_3) &= -3\zeta_3, \\ L_1(-\zeta_1) &= \wp_{2,0}, & L_1(-\zeta_3) &= \wp_{1,3}, \\ L_2(-\zeta_1) &= \frac{1}{2}\wp_{3,0} - \zeta_3 + \frac{3}{5}\lambda_4 u_1, & L_2(-\zeta_3) &= \frac{1}{2}\wp_{2,3} + \frac{4}{5}\lambda_4 \zeta_1 + \left(\frac{4}{5}\lambda_4^2 - 3\lambda_8\right) u_3, \\ L_3(-\zeta_1) &= \wp_{1,3}, & L_3(-\zeta_3) &= \wp_{0,6}, \\ L_4(-\zeta_1) &= \wp_{2,3} + \frac{2}{5}\lambda_6 u_1 - \lambda_8 u_3, & L_4(-\zeta_3) &= \wp_{1,6} + \frac{6}{5}\lambda_6 \zeta_1 - \lambda_4 \zeta_3 - \lambda_8 u_1 + 6\left(\frac{1}{5}\lambda_4 \lambda_6 - \lambda_{10}\right) u_3, \\ L_6(-\zeta_1) &= \frac{1}{2}\wp_{1,6} + \frac{1}{5}\lambda_8 u_1 - 2\lambda_{10} u_3, & L_6(-\zeta_3) &= \frac{1}{2}\wp_{0,9} + \frac{3}{5}\lambda_8 \zeta_1 - 2\lambda_{10} u_1 + \frac{3}{5}\lambda_4 \lambda_8 u_3. \end{aligned}$$

Proof. For the operators L_1, L_3 this result follows from (29), (3) and (4). For the operators L_0, L_2, L_4, L_6 this result follows from (29) and the proof of Theorem 29. □

The following result is based on the described in Theorem 29 action of \mathcal{L}_i , $i = 0, 1, 2, 3, 4, 6$ on the differential generators x_2 and z_4 of the field \mathcal{F}_2 .

Theorem 32. *The commutation relations in the Lie \mathcal{F}_2 -algebra $\text{Der } \mathcal{F}_2$ of derivations of the field \mathcal{F}_2 have the form*

$$\begin{aligned}
[L_0, L_k] &= kL_k, \quad k = 1, 2, 3, 4, 6; & [L_1, L_2] &= \wp_{2,0}L_1 - L_3; \\
[L_1, L_3] &= 0; & [L_1, L_4] &= \wp_{1,3}L_1 + \wp_{2,0}L_3; \\
[L_1, L_6] &= \wp_{1,3}L_3; & [L_2, L_3] &= -\left(\wp_{1,3} + \frac{4}{5}\lambda_4\right)L_1; \\
[L_3, L_4] &= \left(\wp_{0,6} + \frac{6}{5}\lambda_6\right)L_1 + (\wp_{1,3} - \lambda_4)L_3; & [L_3, L_6] &= \frac{3}{5}\lambda_8L_1 + \wp_{0,6}L_3; \\
[L_2, L_4] &= \frac{8}{5}\lambda_6L_0 - \frac{1}{2}\wp_{2,3}L_1 - \frac{8}{5}\lambda_4L_2 + \frac{1}{2}\wp_{3,0}L_3 + 2L_6; \\
[L_2, L_6] &= \frac{4}{5}\lambda_8L_0 - \frac{1}{2}\wp_{1,6}L_1 + \frac{1}{2}\wp_{2,3}L_3 - \frac{4}{5}\lambda_4L_4; \\
[L_4, L_6] &= -2\lambda_{10}L_0 - \frac{1}{2}\wp_{0,9}L_1 + \frac{6}{5}\lambda_8L_2 + \frac{1}{2}\wp_{1,6}L_3 - \frac{6}{5}\lambda_6L_4 + 2\lambda_4L_6.
\end{aligned}$$

Proof. Due to linearity, the relation

$$[L_0, L_k] = kL_k, \quad k = 1, 2, 3, 4, 6,$$

can be checked independently for every summand in the expression (29) for L_k . It is easy to see that it is correct due to the grading.

The expressions for $[L_m, L_n]$, where m or n is equal to 1 or 3, can be obtained from (29) by some simple calculations using Lemmas 30, 31.

It remains to prove the commutation relations between L_2, L_4 , and L_6 . We express $[L_m, L_n]$, where $m < n$; $m, n = 2, 4, 6$, in the form

$$[L_m, L_n] = a_{m,n,0}L_0 + a_{m,n,-1}L_1 + a_{m,n,-2}L_2 + a_{m,n,-3}L_3 + a_{m,n,-4}L_4 + a_{m,n,-6}L_6.$$

We have $\deg a_{i,j,-k} = i + j - k$. By applying both sides of this equation to λ_k and using the explicit expressions for L_k , we get

$$[\ell_m, \ell_n]\lambda_k = (a_{m,n,0}\ell_0 + a_{m,n,-2}\ell_2 + a_{m,n,-4}\ell_4 + a_{m,n,-6}\ell_6)\lambda_k.$$

From this formula and Lemma 30 we obtain the values of the coefficients $a_{m,n,-k}$, $k = 0, 2, 4, 6$:

$$[L_2, L_4] = \frac{8}{5}\lambda_6L_0 + a_{2,4,-1}L_1 - \frac{8}{5}\lambda_4L_2 + a_{2,4,-3}L_3 + 2L_6; \quad (37)$$

$$[L_2, L_6] = \frac{4}{5}\lambda_8L_0 + a_{2,6,-1}L_1 + a_{2,6,-3}L_3 - \frac{4}{5}\lambda_4L_4; \quad (38)$$

$$[L_4, L_6] = -2\lambda_{10}L_0 + a_{4,6,-1}L_1 + \frac{6}{5}\lambda_8L_2 + a_{4,6,-3}L_3 - \frac{6}{5}\lambda_6L_4 + 2\lambda_4L_6. \quad (39)$$

In subsequent calculations we compare the actions of the left and right hand sides of the expressions (37)–(39) on the coordinates u_1 and u_3 . Herewith we use the expressions (27), (29) and Lemma 31.

We provide the calculation of the coefficient $a_{2,4,-1}$. The left hand side of (37) gives

$$\begin{aligned}
[L_2, L_4]u_1 &= L_2(-\zeta_3 + \frac{6}{5}\lambda_6u_3) - L_4(-\zeta_1 + \frac{4}{5}\lambda_4u_3) = \\
&= L_2(-\zeta_3) + \frac{6}{5}\ell_2(\lambda_6)u_3 - \frac{6}{5}\lambda_6u_1 - L_4(-\zeta_1) - \frac{4}{5}\ell_4(\lambda_4)u_3 - \frac{4}{5}\lambda_4(-\zeta_1 - \lambda_4u_3) = \\
&= -\frac{1}{2}\wp_{2,3} + \frac{8}{5}\lambda_4\zeta_1 - \frac{8}{5}\lambda_6u_1 + \frac{2}{5}\left(3\lambda_8 - \frac{16}{5}\lambda_4^2\right)u_3.
\end{aligned}$$

The right hand side of (37) gives

$$[L_2, L_4]u_1 = a_{2,4,-1} + \frac{8}{5}\lambda_4\zeta_1 - \frac{8}{5}\lambda_6u_1 + \frac{2}{5}\left(3\lambda_8 - \frac{16}{5}\lambda_4^2\right)u_3.$$

By equating we obtain $a_{2,4,-1} = -\frac{1}{2}\wp_{2,3}$.

The coefficients $a_{2,4,-3}$, $a_{2,6,-1}$, $a_{2,6,-3}$, $a_{4,6,-1}$, $a_{4,6,-3}$ are calculated analogously. □

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